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An introduction to stochastic integration

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Resum *(CAT)*
L'objectiu d'aquest treball és el d'estudiar integrals estocàstiques que no són necessàriament respecte del moviment brownià.
Primer de tot es revisa la construcció d'aquesta darrera integral per motivar les possibles extensions a altres integradors com són les martingales.
A continuació, estudiem les integrals respecte de camps aleatoris, on comencem per estudiar aquestes integrals respecte del soroll blanc gaussià per, un cop més, estendre la classe d'integradors.
Alhora que es van estudiant aquests objectes, també presentem alguns resultats referents a l'aproximació en llei d'aquests.

Keywords: *Brownian motion, Gaussian process, white noise, martingale, stochastic integral, convergence in law.*

Abstract

The main purpose of this work is to continue and extend the study of the stochastic integral with respect to the Brownian motion usually seen in courses of Stochastic Calculus, providing (hopefully) an introductory text that will allow the average student of the subject (and to anyone who is already familiar with the previously mentioned stochastic integral) to expand his knowledge.

To do so, we briefly review the construction of the Itô integral with respect to the Brownian motion and notice that it turns out that very few features of this particular process are used, which allows us to exploit these ideas to generalize the construction to other processes (mainly, martingales), this is done following the construction provided in the third chapter of [2].

We then discuss the topic of stochastic integration with respect to random fields. We first treat the integral with respect to the space-time Gaussian white noise, following the construction presented in the first two chapters of [1], since it deals with objects which might be a bit more familiar to the intended audience as its construction uses the already studied Itô integral with respect to the Brownian motion and Parseval's identity. Before doing so, we introduce two crucial Gaussian processes (the isonormal process and the white noise), which generalize the Brownian motion and are crucial when it comes to define the stochastic integral with respect to the space-time white noise.

Next, and following the second chapter of [3], we introduce a wider class of random fields (which contains the ones already seen) that can be used as integrators and show how one constructs integrals with respect to such objects. During this process, we use the already studied Gaussian white noise as a canonical example that will serve us as a model to compare the new construction.

Finally, and as we study these objects, we address the problem of how the integrals with respect to the Brownian motion and with respect to the space-time Gaussian white noise can be approximated in law by integrals with respect to random walks. The results obtained, which are motivated by the already known invariance principles like the Donsker's one, can be seen as generalizations of these since the latter can be obtained as a particular case of the former.

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References

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